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2005 J. Phys. A: Math. Gen. 38 721

(http://iopscience.iop.org/0305-4470/38/3/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 721–747 doi:10.1088/0305-4470/38/3/010

NC Calabi–Yau orbifolds in toric varieties with discrete torsion

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Received 20 April 2004, in final form 16 November 2004
Published 23 December 2004
Online at stacks.iop.org/JPhysA/38/721

Abstract

Using the algebraic geometric approach of Berenstein *et al* (hep-th/005087 and hep-th/009209) and methods of toric geometry, we study non-commutative (NC) orbifolds of Calabi–Yau hypersurfaces in toric varieties with discrete torsion. We first develop a new way of getting complex d mirror Calabi–Yau hypersurfaces H_{Δ}^{*d} in toric manifolds $M_{\Delta}^{*(d+1)}$ with a C^{*r} action and analyse the general group of the discrete isometries of H_{Δ}^{*d}. Then we build a general class of d complex dimensional NC mirror Calabi–Yau orbifolds where the non-commutativity parameters $\theta_{\mu\nu}$ are solved in terms of discrete torsion and toric geometry data of $M_{\Delta}^{(d+1)}$ in which the original Calabi–Yau hypersurfaces are embedded. Next we work out a generalization of the NC algebra for generic d-dimensional NC Calabi–Yau manifolds and give various representations depending on different choices of the Calabi–Yau toric geometry data. We also study fractional D-branes at orbifold points. We refine and extend the result for NC $(T^2 \times T^2 \times T^2)/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ to higher dimensional torii orbifolds in terms of Clifford algebra.

PACS numbers: 02.40.Gh, 11.25.−w

1. Introduction

Non-commutative (NC) geometry plays an interesting role in the context of string theory [1] and in compactification of the matrix model formulation of M-theory on NC torii [2–7], which has opened new lines of research devoted to the study of NC quantum field theories [8]; see also [9–22]. In the context of string theory, NC geometry is involved whenever an antisymmetric B-field is turned on. For example, in the study of the ADHM construction $D(p-4)/Dp$ brane systems $(p > 3)$ [23], the NC version of the Nahm construction for monopoles [24] and in the study of tachyon condensation using the so-called GMS approach [25], see also [26–30].

More recently, efforts have been devoted to going beyond the particular NC R^d_θ, NC T^d_θ geometries [28–36]. A special interest has been given to build NC Calabi–Yau manifolds containing the commutative ones as subalgebras and a development has been obtained for the case of orbifolds of Calabi–Yau hypersurfaces. The key point of this construction, using a NC algebraic geometric method [37], see also [38, 39], is based on solving non-commutativity in terms of discrete torsion of the orbifolds. In this regard, there are two ways one may follow to construct this extended geometry: (i) a constrained approach using purely geometric analysis, in which we are interested in this paper, and (ii) crossed product algebra based on the techniques of the fibre bundle and the discrete group representations. For the first method, it has been shown that the $\frac{T^2\times T^2\times T^2}{\mathbf{Z}_2\times\mathbf{Z}_2}$ orbifold of the product of three elliptic curves with torsion, embedded in the C^6 complex space, defines a NC Calabi–Yau threefold [38] having a remarkable interpretation in terms of string states. Moreover, on the fixed planes of this NC threefold, branes fractionate and local complex deformations are no longer trivial. This constrained method was also applied successfully to Calabi–Yau hypersurfaces described by homogeneous polynomials with discrete symmetries including $K3$ and the quintic as particular geometries [38–42]. NC algebraic geometric approach for building NC Calabi–Yau manifolds has very remarkable features and is suspected to have deep connections both with the intrinsic properties of toric varieties [43–45] and the R matrix of Yang–Baxter equations of quantum spaces [46–48].

In this study we extend the Berenstein and Leigh (BL for short) construction for NC Calabi–Yau manifolds with discrete torsion by considering d-dimensional complex Calabi–Yau orbifolds embedded in $(d+1)$ complex toric manifolds and using toric geometry method [49–52]. In particular, we build a general class of d complex dimensional non-commutative mirror Calabi–Yau orbifolds for which the non-commutativity parameters $\theta_{\mu\nu}$ are solved in terms of discrete torsion and toric geometry data of dual polytopes $\Delta(M^d)$. To establish these results, we will proceed in three steps.

(i) We consider pairs of mirror Calabi–Yau hypersurfaces H^d_Δ and H^{*d}_Δ respectively embedded in the toric manifolds M^{d+1}_Δ and $M^{*(d+1)}_\Delta$, where Δ is their attached polyhedron, and develop a manner of handling these spaces by working out the explicit solution for the so-called $Y_\alpha=\prod_{i=1}^{k+1}x_i^{\langle V_i,V^*_\alpha\rangle}$ invariants of the C^{*r} actions and their mirrors $y_i=\prod_{I=1}^{k^*}z_I^{\langle V^*_I,V_i\rangle}$. The construction we will give here is a new one; it is based on pushing further the solution of the Calabi–Yau constraint equations regarding the invariants under the C^{*r} toric actions. Aspects of this analysis may be approached with the analysis of [52, 53], but the novelty is in the manner we treat the C^{*r} invariants. Then we focus our attention on H^{*d}_Δ described by the zero of a homogeneous polynomial $P_\Delta(z)$ of degree D and explore the general form of the group of discrete symmetries Γ of H^{*d}_Δ using the toric geometry data $\{q^a_i;\, V_i;\, 1\leqslant i\leqslant k+1;\, 1\leqslant a\leqslant r;\, d=(k-r)\}$ of the polyhedron Δ.

(ii) We show that for the special region in the moduli space where complex deformations are set to zero, the polynomials P_Δ defining the Calabi–Yau hypersurfaces have a larger group of discrete symmetries Γ_0 containing as a subgroup the usual Γ_{cd} one; $\Gamma_{cd}\subset\Gamma_0$. We treat separately the two corresponding orbifolds $\mathcal{O}_0$ and $\mathcal{O}_{cd}$ and study their link to each other.

(iii) Finally, we construct the NC extension of the Calabi–Yau hypersurfaces by first deriving the right constraint equations, and then solving non-commutativity in terms of discrete torsion and toric geometry data of the variety.

This method can be applied to higher-dimensional NC torii orbifolds extending the result of NC $(T^2\times T^2\times T^2)/(\mathbf{Z}_2\times\mathbf{Z}_2)$ Calabi–Yau threefolds. In this case, the general solution is given in terms of d-dimensional Clifford algebra.

The organization of this paper is as follows: in section 2, we review the main lines of Calabi–Yau hypersurfaces using toric geometry methods. Then we develop a method of getting complex d Calabi–Yau mirror coset manifolds C^{k+1}/C^{*r}, $k-r=d$, as hypersurfaces in WP^{d+1}, by solving the y_i invariants of mirror geometry in terms of invariants of the C^* action of the weighted projective space and the toric geometry data of C^{k+1}/C^{*r}. In section 3, we explore the general form of discrete symmetries of the mirror hypersurface using their toric geometry data. Then we discuss orbifolds of toric Calabi–Yau hypersurfaces. In section 4, we build the corresponding NC toric Calabi–Yau algebras using the algebraic geometry approach of [37, 38]. Then we work out explicitly the matrix realizations of these algebras using toric geometry ideas. In section 5, we give the link with the BL construction while in section 6 we give the generalization of the NC $\frac{T^2\times T^2\times T^2}{\mathbf{Z}_2\times\mathbf{Z}_2}$ orbifold to $\frac{(T^2)^{\otimes(2k+1)}}{\mathbf{Z}_2^{2k}}$, $k\geqslant 1$, where $(T^2)^{\otimes(2k+1)}$ is realized by $(2k+1)$ elliptic curves embedded in $C^{(4k+2)}$ complex space. Our construction, which generalizes that of [38] given by $k=1$, involves non-commuting operators satisfying the $2k$-dimensional Clifford algebra. We end this paper by giving our conclusion.

2. Toric geometry of CY manifolds

2.1. Toric realization of CY manifolds

The simplest $(d+1)$ complex dimensional toric manifold, which we denote as $M_\triangle^{d+1}$, is given by the usual complex projective space $P^{d+1}=\{C^{d+2}-\mathbf{0}_{d+2}\}/C^*$ [54–56]. One can also build $M_\triangle^{d+1}$ varieties by considering the $(k+1)$-dimensional complex spaces C^{k+1}, parametrized by the complex coordinates $\{\mathbf{x}=(x_1,x_2,x_3,\ldots,x_{k+1})\}$, and r toric actions T_a acting on the x_i as

$$T_a : x_i \to x_i\big(\lambda_a^{q_i^a}\big). \tag{2.1}$$

Here the λ_a are r non-zero complex parameters and q_i^a are integers defining the weights of the toric actions T_a. Under these actions, the x_i form a set of homogeneous coordinates defining a $(d+1)$ complex dimensional coset manifold $M^{d+1}=(C^{k+1})/C^{*r}$ with dimension $d=(k-r)$.

More generally, toric manifolds may be thought of as the coset space $(C^{k+1}-\mathcal{P})/C^{*r}$ with $\mathcal{P}$ a given subset of C^{k+1} defined by the C^{*r} action and a chosen triangulation. $\mathcal{P}$ generalizes the standard $\{\mathbf{0}_{k+1}=(0,0,0,\ldots,0)\}$ singlet subset that is removed in the case of P^k. One of the beautiful features of toric manifolds is their nice geometric realization known as the toric geometry representation. The toric data of this realization are encoded in a polyhedron Δ generated by $(k+1)$ vertices carrying all geometric informations on the manifold. These data are stable under C^{*r} actions and are useful in the geometric engineering method of $4D\mathcal{N}=2$ supersymmetric quantum field theory, in particular, in the building of the basic $(d+1)$ gauge invariant coordinate system $\{u_I\}$ of the $(C^{k+1}-\mathcal{P})/C^{*r}$ coset manifold in terms of the homogeneous coordinates x_i [49–52, 54].

In toric geometry, $(d+1)$ complex manifolds $M_\triangle^{d+1}$ are generally represented by an integral polytope Δ spanned by $(k+1)$ vertices V_i of the standard lattice Z^{d+1}. These vertices fulfil r relations given by

$$\sum_{i=1}^{k+1} q_i^a V_i = 0, \qquad a=1,\ldots,r, \tag{2.2}$$

and are in one-to-one correspondence with the r actions of C^{*r} on the complex coordinates x_i (equation (2.1)). In the above relation, the q_i^a integers are the same as in equation (2.1) and are interpreted, in the $\mathcal{N}=2$ gauged linear sigma model language, as the $U(1)^r$ gauge charges of the x_i complex field variables of two-dimensional $\mathcal{N}=2$ chiral multiplets [55–61]. They are also known as the entries of the Mori vectors describing the intersections of complex curves C_a and divisors D_i of $M_{\triangle}^{d+1}$ [62–64].

Submanifolds $\mathcal{N}$ of $M_{\triangle}^{d+1}$ may also be studied by using the Δ toric data $\{q_i^a, V_i\}$ of the original manifold. An interesting example of $M_{\triangle}^{d+1}$ subvarieties is given by the d complex dimensional Calabi–Yau manifolds $H_{\triangle}^d$ defined as hypersurfaces in $M_{\triangle}^{d+1}$ as follows [52]:

$$p(x_1, x_2, x_3, \ldots, x_{k+1}) = \sum_I b_I \prod_{i=1}^{k+1} x_i^{\langle V_i, V_I^* \rangle} = 0, \tag{2.3}$$

together with the Calabi–Yau condition

$$\sum_{i=1}^{k+1} q_i^a = 0, \qquad a = 1, \ldots, r. \tag{2.4}$$

The V_I^* appearing in relation (2.3) are vertices in the dual polytope Δ^* of Δ; their scalar product with the V_i is positive, $\langle V_i, V_I^* \rangle \geqslant 0$. For convenience, we will set from now on $\langle V_i, V_I^* \rangle = n_i^I$. The b_I coefficients are complex moduli describing the complex structure of $H_{\triangle}^d$; their number is given by the Hodge number $h^{(d-1,1)}(H_{\triangle}^d)$. Using the n_i^I integers, the d-dimensional hypersurfaces $H_{\triangle}^d$ in $M_{\triangle}^{d+1}$ (equation (2.3)) read

$$\sum_I b_I \prod_{i=1}^{k+1} x_i^{n_i^I} = 0. \tag{2.5}$$

At this stage it is interesting to make some remarks regarding the above relation. At first sight, one is tempted to make a correspondence between this relation and the hypersurface equation used in [38] and take it as the starting point to build NC Calabi–Yau manifolds *à la* Berenstein *et al*. However, this is not so obvious; first because the polynomial (2.5) is not a homogeneous one and second even though one wants to try to bring it to a homogeneous form, one has to specify the toric data $\{q_I^{*A}; V_I^*\}$ of the polyhedron Δ^*, mirror to $\{q_i^a; V_i\}$ data of Δ. The mirror data satisfy similar relations as (2.2) and (2.4), namely

$$\sum_{I=1}^{k^*+1} q_I^{*A} = 0, \qquad A = 1, \ldots, r^*, \qquad \sum_{I=1}^{k^*+1} q_I^{*A} V_I^* = 0, \qquad A = 1, \ldots, r^*, \tag{2.6}$$

together with $k + 1 - r = k^* + 1 - r^* = d$. Moreover, setting $Y_I = \prod_{i=1}^{k+1} x_i^{n_i^I}$, the above polynomial becomes a linear combination of the Y_I gauge invariants as $\sum_I b_I Y_I = 0$. This relation can however be rewritten in terms of the $(d+1)$-dimensional generator basis $\{Y_\alpha; 1 \leqslant \alpha \leqslant (d+1)\}$ as follows,

$$1 + \sum_{\alpha=1}^{d+1} b_\alpha Y_\alpha + \sum_{I=d+2}^{k^*+1} b_I Y_I = 0, \tag{2.7}$$

where the remaining Y_I invariants, that is the set $\{Y_I; (d+2) \leqslant I \leqslant (k^*+1)\}$, are determined by solving the following Calabi–Yau constraint equations:

$$\prod_{I=1}^{k^*+1} Y_I^{q_I^{*A}} = 1; \qquad A = 1, \ldots, r^*. \tag{2.8}$$

To realize relation (2.7) as a homogeneous polynomial describing the hypersurfaces $H_{\triangle}^d$ with the desired properties, in particular the Calab–Yau condition, one has to solve the above constraint equations. Though this derivation can *a priori* be done using (2.8), we will not proceed in that way. What we will do instead is to use the so-called mirror Calabi–Yau manifolds $H_\triangle^{d*}$ and derive their homogeneous description. The point is that the mirror geometry has some specific features and constraint equations that involve directly the toric data $\{q_i^a; V_i\}$ of the Δ polyhedron contrary to the original hypersurfaces $H_\triangle^d$ which involve the $\{q_I^{*A}; V_I^*\}$ data of Δ^*. Once the rules of getting the $H_\triangle^{d*}$ homogeneous hypersurfaces are defined, one can also reconsider the analysis of $H_\triangle^d$ by starting from relations (2.7) and (2.8), use the Δ^* toric data and perform similar analysis to that we will be developing below.

Under mirror symmetry, toric manifolds $M_\triangle^{(d+1)}$ and Calabi–Yau hypersurfaces $H_\triangle^d$ are mapped to $M_\triangle^{(d+1)*}$ and $H_\triangle^{d*}$ respectively. They are obtained by exchanging the roles of complex and Kahler structures in agreement with the Hodge relations

$$h^{(d-1,1)}\big(H_\triangle^d\big) = h^{(1,1)}\big(H_\triangle^{d*}\big), \qquad h^{(1,1)}\big(H_\triangle^d\big) = h^{(d-1,1)}\big(H_\triangle^{d*}\big), \tag{2.9}$$

and similarly for $M_\triangle^{(d+1)}$ and $M_\triangle^{(d+1)*}$ [63–66]. In practice, the building of $M_\triangle^{(d+1)*}$ and so $H_\triangle^{d*}$ is achieved by using the vertices V_I^* of the convex hull spanned by the V_α^*. Following [65–71], mirror Calabi–Yau manifolds $H_\triangle^{d*}$ are given by the zero of the polynomial

$$p(z_1, z_2, \ldots, z_{k^*+1}) = \sum_{i=1}^{k+1} a_i \prod_{I=1}^{k^*+1} \big(z_I^{n_i^I}\big), \tag{2.10}$$

where the z_I are the mirror coordinates. The C^{*r^*} actions of $M_\triangle^{(d+1)*}$ act on the z_I as

$$z_I \to z_I \lambda_I^{q_I^{*A}}, \tag{2.11}$$

with q_I^{*A} as in equation (2.6). The a_i are the complex structure of the mirror Calabi–Yau manifold $H_\triangle^{d*}$; they also describe the Kahler deformations of $H_\triangle^d$. An interesting feature of relation (2.10) is its representation in terms of the $(k+1)$ invariants $y_i = \prod_{I=1}^{k^*+1}\big(z_I^{m_i^I}\big)$ under the C^{*r^*} actions of $M_\triangle^{d*}$, i.e.

$$\sum_{i=1}^{k+1} a_i y_i = 0, \tag{2.12}$$

together with the following r constraint equations of the mirror geometry:

$$\prod_{i=1}^{k+1} \big(y_i^{q_i^a}\big) = 1, \qquad a = 1, \ldots, r. \tag{2.13}$$

These equations involve $(k+1)$ variables y_i; not all of them are independent since they are subject to $(r+1)$ conditions (r from equations (2.13) and one from (2.12)) leading indeed to the right dimension of $H_\triangle^{d*}$. Equations (2.12) and (2.13) will be our starting point towards building NC Calabi–Yau manifolds using the Berenstein *et al* approach. Before that let us put these relations into a more convenient form.

2.2. *Solving the mirror constraint equations*

As shown in the above equations, not all the y_i are independent variables, only $(d+1)$ of them are. In what follows we shall fix this redundancy by using a coordinate patch of the

$(d+1)$ weighted projective spaces WP^{d+1} parametrized by the system of variables $\{u_\alpha, 1 \leqslant \alpha \leqslant (d+1); u_{d+2}\}$. In the coordinate patch $u_{d+2} = 1$, the u_α variables behave as $(d+1)$ independent gauge invariants parametrizing the coset manifold $[(C^{d+2})/C^*] \sim [(C^{k+1})/C^{*r}]$. The remaining r y_i are given by monomials of the u_α. A nice way of getting the relation between y_i and u_α is inspired from the analysis [52, 53]; it is based on introducing the following system $\{N_i; 1 \leqslant i \leqslant (k+1)\}$ of $(d+1)$-dimensional vectors of integer entries $(N_i)_\alpha = \langle V_i, V_\alpha^*\rangle \equiv n_i^\alpha$. From equation (2.2), it is not difficult to see that

$$\sum_{i=1}^{k+1} q_i^a N_i = 0, \qquad a = 1, \ldots, r; \qquad \alpha = 1, \ldots, d+1, \tag{2.14}$$

or equivalently

$$\sum_{i=1}^{k+1} q_i^a n_i^\alpha = 0, \qquad a = 1, \ldots, r; \qquad \alpha = 1, \ldots, d+1. \tag{2.15}$$

Note that the introduction of the system $\left\{(N_i)_\alpha \equiv n_i^\alpha; 1 \leqslant i \leqslant (k+1)\right\}$ has a remarkable interpretation; it describes the complex deformations of $H_\triangle^{d*}$ and by the correspondence (2.9) the Kahler ones of $H_\triangle^d$. Observe also that shifting the N_i by a constant vector, say t_0, equation (2.14) remains invariant due to the Calabi–Yau condition (2.4). Therefore the V_i vertices of equations (2.2) can be solved by a linear combination of N_i and t_0; $V_i = N_i + at_0$. Having these relations in mind, we can use them to reparametrize the y_i invariants in terms of the $(d+2)$ generators $u_\mu (u_{d+2}$ arbitrary) as follows:

$$y_i = u_1^{(n_i^1-1)} u_2^{(n_i^2-1)} \cdots u_{d+1}^{(n_i^{d+1}-1)} u_{d+2}^{(n_i^{d+2}-1)} = \prod_{\mu=1}^{d+2} u_\alpha^{(n_i^\mu-1)}, \tag{2.16}$$

$$y_0 = 1 \quad \Leftrightarrow \quad \left(n_0^\alpha - 1\right) = 0, \qquad \forall\, \alpha = 1, \ldots, d+2. \tag{2.17}$$

Note that $\prod_{i=1}^{k+1}\left(y_i^{q_i^a}\right) = 1$ is automatically satisfied due to equations (2.14) and (2.15). Note also the n_i^{d+2} integers are extra quantities introduced for later use; they should not be confused with the $\left\{n_i^\alpha; 1 \leqslant \alpha \leqslant d+1\right\}$ entries of N_i. Putting relations (2.16) and (2.17) back into equation (2.12), we get an equivalent way of writing equation (2.10), namely

$$a_0 1 + \sum_{i=1}^{k+1} a_i u_1^{(n_i^1-1)} u_2^{(n_i^2-1)} \cdots u_{d+1}^{(n_i^{d+1}-1)} u_{d+2}^{(n_i^{d+2}-1)} = 0. \tag{2.18}$$

The main difference between this relation and equation (2.10) is that the above one involves $(d+2)$ variables only, in contrast to the case of equation (2.10) which rather involves $(d+r^*+1)$ coordinates; that is r^* variables more. Equation (2.18) is then a relation where the C^{*r^*} symmetries on the z_I equation (2.11) are completely fixed. Indeed starting from equation (2.10), it is not difficult to rederive equation (2.18) by working in the remarkable coordinate patch $\mathcal{U} = \{(z_1, z_2, \ldots, z_{d+2,} 1, 1, \ldots, 1)\}$, which is isomorphic to a weighted projective space $WP^{d+1}_{(\delta_1,\ldots,\delta_{d+2})}$ with a weight vector $\delta_\mu = (\delta_1, \ldots, \delta_{d+2})$. In this way of viewing things, the y_i variables may be thought of as gauge invariants under the projective action $WP^{d+1}_{(\delta_1,\ldots,\delta_{d+2})}$ and consequently the Calabi–Yau manifold (2.18) as a hypersurface in $WP^{d+1}_{(\delta_1,\ldots,\delta_{d+2})}$ described by a homogeneous polynomial $p(u_1, \ldots, u_{d+2})$ embedded of degree $D = \sum_{\mu=1}^{d+2} \delta_\mu$. Thus, under the projective action $u_\mu \longrightarrow \lambda^{\delta_\mu} u_\mu$, the monomials $y_i = \prod_{\mu=1}^{d+2}\left(u_\mu^{(n_i^\mu-1)}\right)$ transform as $y_i \lambda^{\sum_\mu (\delta_\mu (n_i^\mu - 1))}$ and so the following constraint equations should hold:

$$\sum_{\mu=1}^{d+2} \delta_\mu = D, \tag{2.19}$$

$$\sum_{\mu=1}^{d+2} \delta_\mu n_i^\mu = D. \tag{2.20}$$

These relations show that the n_i^μ integers can be solved in terms of the partitions d_i^μ of the degree D of the homogeneous polynomial $p(u_1, \ldots, u_{d+2})$. Indeed from $\sum_{\mu=1}^{d+2} d_i^\mu = D$, one sees that $n_i^\mu = \frac{d_i^\mu}{\delta_\mu}$, among which we have the following remarkable ones:

$$n_i^\mu = \frac{D}{\delta_\mu} \qquad \text{if} \quad i = \mu \quad \text{for} \quad 1 \leqslant \mu \leqslant d+2. \tag{2.21}$$

To get the V_i vertices, we keep the $\left\{n_i^\alpha; 1 \leqslant \alpha \leqslant d+1\right\}$ entries and subtract the trivial monomial associated with $\left\{\left(t_0^\alpha\right) = (1, 1, \ldots, 1)\right\}$. So the V_i vertices are

$$V_i^\alpha = n_i^\alpha - t_0^\alpha = \frac{d_i^\alpha}{\delta_\alpha} - t_0^\alpha. \tag{2.22}$$

For the $(d+3)$ leading vertices, we have

$$\begin{aligned}
V_0 &= (0, 0, 0, \ldots, 0, 0)\\
V_1 &= \left(\frac{D}{\delta_1} - 1, -1, -1, \ldots, -1, -1\right),\\
V_2 &= \left(-1, \frac{D}{\delta_2} - 1, -1, \ldots, -1, -1\right)\\
V_3 &= \left(-1, -1, \frac{D}{\delta_3} - 1, \ldots, -1, -1\right)\\
&\vdots\\
V_{d+1} &= \left(-1, -1, -1, \ldots, \frac{D}{\delta_{d+1}} - 1, -1\right),\\
V_{d+2} &= \left(-1, -1, -1, \ldots, -1, \frac{D}{\delta_{d+2}} - 1\right).
\end{aligned} \tag{2.23}$$

Before going ahead, let us give some remarks: (a) the integrality of the entries of these vertices requires that the D degree should be a common multiple of the weights δ_μ. Moreover, the number of partitions of D should be less than $(k+2)$. (b) As far as the $(d+3)$ leading vertices are concerned, the corresponding homogeneous monomials are

$$N_0 \to \prod_{\mu=1}^{d+2} u_\mu, \tag{2.24}$$

$$N_\mu \longrightarrow u_\mu^{\frac{D}{\delta_\mu}}, \qquad \mu = 1, \ldots, d+2. \tag{2.25}$$

So the corresponding mirror polynomial takes the form

$$\sum_{\mu=1}^{d+2} u_\mu^{\frac{D}{\delta_\mu}} + a_0 \prod_{\mu=1}^{d+2} (u_\mu) = 0. \tag{2.26}$$

More generally, the mirror polynomial $P_\Delta(u)$ describing H_Δ^{d*} reads

$$P_\Delta(u) = \sum_{\mu=1}^{d+2} u_\mu^{\frac{D}{\delta_\mu}} + a_0 \prod_{\mu=1}^{d+2} (u_\mu) + \sum_{i=d+3}^{k+1} a_i \prod_{\mu=1}^{d+2} \left(u_\mu^{n_\mu^i}\right) = 0, \tag{2.27}$$

where the a_i are complex moduli of the mirror Calabi–Yau hypersurface.

2.3. More on the mirror CY geometry

Here we further explore the relations between the realizations (2.10) and (2.27) of the mirror Calabi–Yau manifolds. In particular, we give an explicit derivation of the weights δ_μ involved in the polynomials (2.27) in terms of the Calabi–Yau q_i^a charges. To do so, first of all recall that under the projective action

$$u_\mu \longrightarrow \lambda^{\delta_\mu} u_\mu, \tag{2.28}$$

the polynomial $P_\Delta(u)$ behaves as $P_\Delta(\lambda^{\delta_\mu} u) = \lambda^D P_\Delta(u)$ leaving the zero locus invariant. Using the identity $\sum_{\mu=1}^{d+2} \delta_\mu = D$, one may reinterpret the Calabi–Yau condition (2.4) or equivalently by introducing r integers p_a

$$\sum_{\mu=1}^{d+2} \sum_{a=1}^{r} p_a q_\mu^a = - \sum_{i=d+3}^{k+1} \sum_{a=1}^{r} p_a q_i^a,$$

by thinking about it as

$$\delta_\mu = \sum_{a=1}^{r} p_a q_\mu^a \tag{2.29}$$

$$D = \sum_{i=d+3}^{k+1} \delta_i = - \sum_{i=d+3}^{k+1} \sum_{a=1}^{r} p_a q_i^a. \tag{2.30}$$

For instance, for ordinary projective spaces P^k, we can use the generalization of the transformation introduced in [38], namely

$$u_\mu \longrightarrow \omega^{Q_\mu^a} u_\mu, \qquad u_{d+2} \longrightarrow u_{d+2}, \tag{2.31}$$

where, roughly speaking, ω is a Dth root of unity. This transformation leaves $P_\Delta(u)$ invariant as far as the Q_μ^a obey the Calabi–Yau condition $\sum_{\mu=1}^{d+1} Q_\mu^a = 0$ and $Q_{d+2}^a = 0$, in agreement with the choice of the coordinate patch $u_{d+2} = 1$. Next by appropriate choice of λ, we can compare both the transformations (2.28) and (2.31) as well as their actions on the monomials $y_i = \prod_{\mu=1}^{d+2} \left(u_\mu^{(n_i^\mu - 1)}\right)$ respectively given by $y_i \longrightarrow y_i \omega^{\sum_\mu \delta_\mu (n_i^\mu - 1)}$ and $y_i \longrightarrow y_i \omega^{\sum_\mu Q_\mu^a (n_i^\mu - 1)}$. Invariance under these actions leads to equations (2.19) and (2.20), and their toric geometry equations analogue

$$\sum_{\mu=1}^{d+2} Q_\mu^a = 0 \quad \text{modulo } (D) \tag{2.32}$$

$$\sum_{\mu=1}^{d+2} Q_\mu^a n_i^\mu = 0 \quad \text{modulo } (D). \tag{2.33}$$

Comparing these equations with equations (2.32)–(2.33) and (2.19)–(2.20), one gets the following relation between the Q_μ^a and q_i^a charges of the original manifold:

$$Q_\mu^a = \left(q_\mu^a + \frac{1}{d+2}\sum_{i=d+3}^{k} q_i^a\right) \quad \text{modulo } (D). \tag{2.34}$$

As the isometries of equations (2.26) and (2.27) will be involved in the study of the NC hypersurface Calabi–Yau orbifolds, let us derive a general form of these isometries using geometry toric data. We will distinguish between two cases: (i) the group of isometries Γ_0 leaving equation (2.26) invariant and (ii) its subgroup Γ_{cd} of discrete symmetries of equation (2.27) commuting with complex deformations.

3. Discrete symmetries and CY orbifolds

To determine the discrete symmetries of the Calabi–Yau homogeneous hypersurfaces, let us derive the general groups Γ_0 and Γ_{cd} of transformations leaving equations (2.26) and (2.27) invariant:

$$\Gamma = \{g_\omega \mid g_W : u_\mu \to g_\omega(u_\mu) = u'_\mu = u_\mu(\mathcal{W})^{\mathbf{b}_\mu};\ P_\Delta(u') = P_\Delta(u)\}, \tag{3.1}$$

where

$$\mathcal{W}^{\mathbf{b}_\mu} = \prod_{\nu=1}^{d+2}\left[(\omega_\nu)^{\mathbf{a}_\mu^\nu}\right]$$

and where $\{\mathbf{b}_\mu\}_{1\leqslant\mu\leqslant d+2}$ is a $(d+2)$-dimensional vector weight and $\mathbf{a}_\mu^\nu$ are their entries. They will be determined by symmetry requirements and the Calabi–Yau toric geometry data. As the solutions we will build depend on the weights δ_μ, we will distinguish hereafter the P^{d+1} and WP^{d+1} spaces, a matter of illustrating the idea and the techniques we will be using.

3.1. P^{d+1} projective spaces

The crucial point to note here is that because of the equality $\delta_1 = \delta_2 = \cdots = \delta_{d+2} = 1$, the D degree of the polynomials $P_\Delta(u)$ is equal to $(d+2)$ and so the constraint equation (2.20) reduces to $\sum_{\mu=1}^{d+2} n_i^\mu = (d+2)$ for any value of the i index. Putting back $\delta_\mu = 1$ in equations (2.26), one sees that invariance under Γ_0 of the first terms u_μ^{d+2} shows that a natural solution is given by taking $\omega_1 = \omega_2 = \cdots = \omega_{d+1} = \omega = \exp \mathrm{i}\left(\frac{2\pi}{d+2}\right)$ and then $\omega^{\mathbf{b}_\mu} = \exp \mathrm{i}\frac{2\pi}{d+2}\mathbf{b}_\mu$. However, invariance of the term $\prod_{\mu=1}^{d+2}(u_\mu)$ under the change (3.1), implies that $\mathbf{b}_\mu$ should satisfy the following constraint equation:

$$\sum_{\mu=1}^{d+2} \mathbf{b}_\mu = 0, \quad \text{modulo } (d+2). \tag{3.2}$$

In what follows, we shall give an explicit class of special solutions for the constraint equation $\sum_{\mu=1}^{d+2}\mathbf{b}_\mu = 0$, by using the toric geometry data of the H_Δ^d Calabi–Yau manifold equations (2.2) and (2.4). The solutions, modulo $(d+2)$, are obtained by making appropriate shifts.

3.1.1. Explicit construction of $\mathbf{b}_\mu$ weights. The solution for $\mathbf{b}_\mu$ we will construct below contains two terms which are intimately linked to toric geometry equations (2.2) and (2.4). To have an idea of the explicit derivation of the $\mathbf{b}_\mu$, let us first introduce the following two Q_μ and ξ_μ quantities. They will be used in realizing $\mathbf{b}_\mu$.

The Q_μ weights. This is a quantity defined as

$$Q_\mu = Q_\mu(p_1, \ldots, p_r) = \sum_{a=1}^{r} p_a Q_\mu^a, \qquad 1 \leqslant \mu \leqslant d+2, \tag{3.3}$$

where the p_a are given integers and Q_μ^a are a kind of shifted Calabi–Yau charges, which they are given in terms of the q_μ^a Mori vectors of the toric manifold shifted by constant numbers τ^a, as shown in the following relation:

$$Q_\mu^a = q_\mu^a + \tau^a. \tag{3.4}$$

The τ^a are determined by requiring that the Q_μ^a shifted charges have to satisfy the Calabi–Yau condition $\sum_{\mu=1}^{d+2} Q_\mu^a = 0$. Using (2.4), we find

$$\tau^a = \frac{1}{d+2} \sum_{i=d+3}^{k+1} q_i^a. \tag{3.5}$$

Replacing Q_μ^a by its explicit expression in terms of the Mori vector charges, we get

$$Q_\mu = \sum_{a=1}^{r} p_a \left(q_\mu^a + \frac{1}{d+2} \sum_{i=d+3}^{k} q_i^a \right). \tag{3.6}$$

It satisfies identically the property $\sum_{\mu=1}^{d+2} Q_\mu = 0$, which we will interpret as the Calabi–Yau condition because of its link with the original relation $\sum_{i=1}^{k+1} q_i^a = 0$.

The ξ_μ weights. These weights carry information on the data of the polytope Δ of the toric varieties and so on their Calabi–Yau submanifolds. They are defined as

$$\xi_\mu = \xi_\mu(s_1, \ldots, s_{d+1}) = \sum_{\alpha=1}^{d+1} s_\alpha \xi_\mu^\alpha \tag{3.7}$$

where the s_α are integers and ξ_μ^α are defined in terms of the toric data of M_Δ^{d+1} as follows:

$$\xi_\mu^\alpha = \sum_{a=1}^{r} p_a \left(q_\mu^a n_\mu^\alpha + \frac{1}{d+2} \sum_{i=d+3}^{k+1} q_i^a n_i^\alpha \right). \tag{3.8}$$

As for the Q_μ weights, one can check here also that the sum $\sum_{\mu=1}^{d+2} \xi_\mu$ vanishes identically due to the constraint equation (2.15).

The $\mathbf{b}_\mu$ weights. A class of solutions for $\mathbf{b}_\mu$ based on the Calabi–Yau toric geometry data (2.2) and (2.4) may be given by a linear combination of the Q_μ and ξ_μ weights as shown below:

$$\mathbf{b}_\mu = m_1 Q_\mu + m_2 \xi_\mu, \tag{3.9}$$

where m_1 and m_2 are integers modulo $(d+2)$. Moreover, setting $\mathbf{b}_\mu = \sum_{\nu=1}^{d+2} \mathbf{a}_\mu^\nu$ and

$$\begin{aligned} Q_\mu^\alpha &= Q_\mu^a \qquad \text{for} \quad \alpha = 1, \ldots, r; \\ Q_\mu^\alpha &= 0 \qquad\ \ \text{for} \quad \alpha = (r+1), \ldots, (d+2), \end{aligned} \tag{3.10}$$

while $Q_\mu^\alpha = Q_\mu^a$ for $r \geqslant d+1$, we can rewrite the above solutions as follows:

$$\mathbf{a}_\mu^\nu = m_1 Q_\mu^\nu + m_2 \xi_\mu^\nu. \tag{3.11}$$

Therefore, the general transformations of the $\Gamma_0(P^{d+1})$ group of discrete isometries are given by the change (3.1) with $\mathbf{b}_\mu$ vector weights depending on $(r+d+1) = k$ integers, namely r integers p_a and $(d+1)$ integers s_α.

3.1.2. Complex deformations. To get the discrete symmetries of the full Calabi–Yau homogeneous complex hypersurface including the complex deformation equation (2.27), one should solve more complicated constraint relations which we give hereafter. Under Γ_{cd} of transformation equation (2.27), the complex deformations of the Calabi–Yau manifold $P_\Delta(u)$ are stable provided the $\mathbf{b}_\mu$ weights satisfy equation (3.2) but also the following constraint equations:

$$\sum_{\mu=1}^{d+2} \mathbf{b}_\mu n_\mu^\nu = 0, \tag{3.12}$$

where the n_μ^ν are as in equation (2.27). A particular solution of these constraint equations is given by taking $\mathbf{b}_\mu = Q_\mu$ that is $m_1 = 1$ and $m_2 = 0$. Indeed replacing $\mathbf{b}_\mu$ by its expression (3.9) and putting back into the above relation, we get with the help of the identity (2.20),

$$\begin{aligned}\left[\sum_{\mu=1}^{d+2}\sum_{a=1}^{r} p_a\big(q_\mu^a + \tau^a\big)n_\mu^\nu\right] &= \sum_{a=1}^{r} p_a \left[\sum_{\mu=1}^{d+2} q_\mu^a n_\mu^\nu + (d+2)\tau^a\right] \\ &= \sum_{a=1}^{r} p_a \left[\sum_{\mu=1}^{d+2} q_\mu^a n_\mu^\nu + \sum_{i=d+3}^{k} q_i^a n_\mu^\nu\right] = 0.\end{aligned} \tag{3.13}$$

For $m_1, m_2 \neq 0$, the relation $\mathbf{b}_\mu = m_1 Q_\mu + m_2 \xi_\mu$ ceases to be a solution of the constraint equation (3.12). Therefore Γ_{cd} is a subgroup of Γ_0. It depends on the p_a integers and involves the Calabi–Yau condition only.

3.2. WP^{d+1} weighted projective spaces

The previous analysis made for the case of P^{d+1} applies as well for WP^{d+1}. Starting from equation (2.26) and making the change (3.1), invariance requirement leads to take the ω_μ group parameters as $\omega_\mu = \exp \mathrm{i}\frac{2\pi\delta_\mu}{D}$ and the $\mathbf{a}_\mu^\nu$ coefficients constrained as

$$\begin{aligned}&\sum_{\nu=1}^{d+2} \delta_\nu \mathbf{a}_\mu^\nu = 0, \quad \text{modulo } \delta_\mu \\ &\sum_{\mu=1}^{d+2} \mathbf{a}_\mu^\nu = 0.\end{aligned} \tag{3.14}$$

Following the same reasoning as before, one can write down a class of solutions, with integer entries, in terms of the previous weights as follows,

$$\mathbf{a}_\mu^\nu = (\delta^\nu)^{-1}\big[m_1 Q_\mu^\nu + m_2 \xi_\mu^\nu\big], \tag{3.15}$$

where Q_μ^ν and ξ_μ^ν are as in equation (3.11). In case where the complex deformations of equation (2.27) are taken into account, the discrete symmetry group is no longer the same since the constraint equation (3.13) is now replaced by the following one:

$$\sum_{\mu=1}^{d+2} \mathbf{a}_\mu^\nu n_\mu^i = 0, \qquad \forall \nu = 1, \ldots, (d+2). \tag{3.16}$$

As in the projective case where the δ_μ are equal to 1, the solutions for the $\mathbf{a}_\mu^\nu$ integers are given by equation (3.15) with $m_1 \neq 0$ and $m_2 = 0$. To conclude this section, one should note that the group of discrete isometries $\Gamma_{cd} \subset \Gamma_0$ of the Calabi–Yau hypersurfaces including complex deformations is intimately related to the Calabi–Yau condition.

4. NC toric CY manifolds

Before revealing our results regarding NC toric Calabi–Yau's, let us begin this section by reviewing briefly the BL idea of building NC orbifolds of Calabi–Yau hypersurface.

4.1. Algebraic geometric approach for CY

Roughly speaking, given a d-dimensional Calabi–Yau manifold X^d described algebraically by a complex equation $p(z_i) = 0$ with a group Γ of discrete isometries. We take quotient of X^d by the action of the finite group Γ

$$\Gamma : z_i \to g z_i g^{-1}, \qquad g \in \Gamma \tag{4.1}$$

such that the following two conditions are fulfilled: $p(z_i)$ polynomial and the $(d, 0)$ holomorphic form are invariants. The latter condition is the equivalent of the vanishing of the first Chern class $c_1 = 0$. Using the discrete torsion, one can build the NC extensions of the orbifold, $\left(\frac{X^d}{\Gamma}\right)_{\mathrm{nc}}$, as follows. The coordinates z_i are replaced by matrix operators Z_i satisfying

$$Z_i Z_j = \theta_{ij} Z_j Z_i. \tag{4.2}$$

Invariance of $p(z_i)$ requires the parameters θ_{ij} to be in the discrete group Γ. Moreover, the Calabi–Yau condition imposes the extra constraint equation

$$\prod_i \theta_{ij} = 1, \qquad \forall j \neq i. \tag{4.3}$$

In this case of the quintic, embedded in a P^5 projective space described by the homogeneous polynomial $p(z_1, \ldots, z_5)$ of degree 5:

$$p(z_i) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 + a_0 \prod_{1=1}^{5} z_i = 0. \tag{4.4}$$

The group Γ acts as $z_i \longrightarrow z_i \omega^{Q_i^a}$ where $\omega^5 = 1$ and the Q_i^a vectors are

$$Q_i^1 = (1, -1, 0, 0, 0), \qquad Q_i^2 = (1, 0, -1, 0, 0), \qquad Q_i^3 = (1, 0, 0, -1, 0) \tag{4.5}$$

In the coordinate patch $\mathcal{U} = \{(z_1, z_2, z_3, z_4); z_5 = 1\}$, equation (4.5) reduces to

$$1 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + a_0 \prod_{j=1}^{4} z_j = 0. \tag{4.6}$$

The local NC algebra $\mathcal{A}_{\mathrm{nc}}$ describing the NC version of equation (4.5) is obtained by associating with z_5 the matrix $z_5 I_5$ and with each holomorphic variable z_i a 5×5 matrix Z_i satisfying the BL algebra

$$\begin{aligned} &Z_1 Z_2 = \alpha Z_2 Z_1, && Z_1 Z_3 = \alpha^{-1} \beta Z_3 Z_1, \\ &Z_1 Z_4 = \beta^{-1} Z_4 Z_1, && Z_2 Z_3 = \alpha\gamma Z_3 Z_2, \\ &Z_2 Z_4 = \gamma^{-1} Z_4 Z_2, && Z_3 Z_4 = \beta\gamma Z_4 Z_3, \end{aligned} \tag{4.7}$$

where α, β and γ are fifth roots of unity. The centre of this algebra $\mathcal{Z}(\mathcal{A}_{\mathrm{nc}}) = \left\{I_5, Z_\nu^5, \prod_{\nu=1}^4 Z_\nu\right\}$, that is,

$$\left[Z_\mu, Z_\nu^5\right] = 0, \qquad \left[Z_\mu, \prod_{\nu=1}^{4} Z_\nu\right] = 0. \tag{4.8}$$

According to the Schur lemma, one can set $Z_\nu^5 = I_5 z_\nu^5$ and $\prod_{\nu=1}^4 Z_\nu = I_5 \prod_{\nu=1}^4 z_\nu$ and so the centre coincides with the equation of the quintic. In what follows we extend this analysis to NC toric Calabi–Yau orbifolds.

4.2. NC toric CY orbifolds

Following the same lines as [37–41] and using the discrete symmetry group Γ, one can build the orbifolds $\mathcal{O} = H_{\triangle}^{d*}\big/\Gamma$ of the Calabi–Yau hypersurface and work out their non-commutative extensions $\mathcal{O}_{\rm nc}$. The main steps in the building of $\mathcal{O}_{\rm nc}$ may be summarized as follows: first start from the Calabi–Yau hypersurfaces $H_{\triangle}^{d*}$ (equations (2.26)–(2.27)) and fix a coordinate patch of WP^{d+1}, say $u_{d+2} = 1$. Then impose the identification under the discrete automorphisms (3.1) defining $H_{\triangle}^{d*}\big/\Gamma$. The NC extension of this orbifold is obtained as usual by extending the commutative algebra $\mathcal{A}_{\rm c}$ of functions on $H_{\triangle}^{d*}\big/\Gamma$ to a NC one $\mathcal{A}_{\rm nc} \sim \mathcal{O}_{\rm nc}$. In this algebra, the u_μ coordinates are replaced by matrix operators U_μ satisfying the algebraic relations

$$U_\mu U_\nu = \theta_{\mu\nu} U_\nu U_\mu, \qquad \nu > \mu = 1, \ldots, d+1, \tag{4.9}$$

where the $\theta_{\mu\nu}$ non-commutativity parameters obey the following constraint relations:

$$\theta_{\mu\nu}\theta_{\nu\mu} = 1, \tag{4.10}$$

$$(\theta_{\mu\nu})^{\frac{D}{\delta_\nu}} = 1, \tag{4.11}$$

$$\prod_{\mu=1}^{d+1}(\theta_{\mu\nu}) = 1, \tag{4.12}$$

as far as equation (2.26) is concerned that is in the region of the moduli space where the complex moduli a_i are zero ($i = 1, \ldots$). However, in the general case where the a_i are non-zero we should have moreover

$$\prod_{\mu=1}^{d+1}\left(\theta_{\mu\nu}^{n_\mu^\alpha}\right) = 1, \qquad \alpha = 1, \ldots, d+1. \tag{4.13}$$

Let us comment briefly on these constraint relations. Equation (4.11) reflects that the parameters $\theta_{\nu\mu}$ are just the inverse of $\theta_{\mu\nu}$ and can be viewed as describing deformations away from the identity suggesting by the occasion that they may be realized as

$$\theta_{\mu\nu} = \exp \eta_{\mu\nu},$$

where $\eta_{\mu\nu} = -\eta_{\nu\mu}$ is the infinitesimal version of the non-commutativity parameter. The constraint (4.12)–(4.13) reflects just the remarkable property according to which $U_\nu^{\frac{D}{\delta_\nu}}$ and $\prod_{\mu=1}^{d+2}(U_\mu)$ are elements in the centre $\mathcal{Z}(\mathcal{A}_{\rm nc})$ of the non-commutative algebra $\mathcal{A}_{\rm nc}$, i.e.

$$\left[U_\mu, U_\nu^{\frac{D}{\delta_\nu}}\right] = 0, \tag{4.14}$$

$$\left[U_\mu, \prod_{\nu=1}^{d+2}(U_\nu)\right] = 0. \tag{4.15}$$

Finally, the constraint equations (4.14), obtained by requiring $\left[U_\mu, \prod_{\nu=1}^{d+2}\left(U_\nu^{n_\mu^\alpha}\right)\right] = 0$, describe the compatibility between non-commutativity and deformations of the complex structure of the Calabi–Yau hypersurfaces.

In what follows we shall solve the above constraint equations (4.11)–(4.14) in terms of toric geometry data of the toric variety in which the mirror geometry is embedded. Since these solutions depend on the weight vector δ we will consider two cases: $\delta_\mu = 1$ for all values of μ and δ_μ taking general numbers (equations (2.20)).

4.2.1. Matrix representations for projective spaces. The analysis we have developed so far can be made more explicit by computing the NC algebras associated with the Calabi–Yau hypersurface orbifolds with discrete torsion. In this regard, a simple and instructive class of solutions of the above constraint equations may be worked in the framework of the P^{d+1} ordinary projective spaces. To do this, consider a d complex dimensional Calabi–Yau homogeneous hypersurfaces in P^{d+1}, namely,

$$u_1^{d+2} + u_2^{d+2} + u_3^{d+2} + u_4^{d+2} + \cdots + u_{d+2}^{d+2} + a_0 \prod_{\mu=1}^{d+2} u_\mu = 0, \tag{4.16}$$

with the discrete isometries (2.31) and Calabi–Yau charges Q_μ^a satisfying

$$\sum_{\mu=1}^{d+2} Q_\mu^a = 0, \qquad a = 1, \ldots, d. \tag{4.17}$$

From constraint equation (4.12), it is not difficult to see that $\theta_{\mu\nu}$ is an element of the $\mathbf{Z}_{d+2}$ group and so can be written as

$$\theta_{\mu\nu} = \omega^{L_{\mu\nu}}, \tag{4.18}$$

where $\omega = \exp \frac{2\pi \mathrm{i}}{d+2}$ and $L_{\mu\nu}$ is a $(d+1) \times (d+1)$ antisymmetric matrix, i.e. $L_{\mu\nu} = -L_{\nu\mu}$, as required by equation (4.11). Putting this solution back into equation (4.13), one discovers that this tensor should satisfy

$$\sum_{\mu=1}^{d+1} L_{\mu\nu} = 0, \quad \text{modulo } (d+2). \tag{4.19}$$

Using the toric data of the Calabi–Yau manifold $\sum_{\mu=1}^{d+1} Q_\mu^a = 0$ and $\sum_{\mu=1}^{d+1} \xi_\mu^\alpha = 0$, namely

$$Q_\mu = \sum_{a=1}^{r} p_a \left(q_\mu^a + \frac{1}{d+1} \sum_{i=d+2}^{k} q_i^a \right), \tag{4.20}$$

$$\xi_\mu^\alpha = \sum_{a=1}^{r} p_a \left(q_\mu^a n_\mu^\alpha + \frac{1}{d+1} \sum_{i=d+2}^{k+1} q_i^a n_i^\alpha \right), \tag{4.21}$$

one sees that the $L_{\mu\nu}$ can be solved as bilinear forms of Q_μ^a and ξ_μ^α, namely

$$L_{\mu\nu} = L_1 \Omega_{ab} Q_\mu^a Q_\nu^b + L_2 \Omega_{\alpha\beta} \xi_\mu^\alpha \xi_\nu^\beta. \tag{4.22}$$

Here L_1 and L_2 are numbers modulo $(d+2)$ and Ω_{ab} and $\Omega_{\alpha\beta}$ are respectively the antisymmetric $r \times r$ and $(d+2) \times (d+2)$ for even integer values of r and d or their generalized expressions otherwise. Moreover, $L_{\mu\nu}$ can also be rewritten in terms of the $\mathbf{a}_\mu^\nu$ components of $\mathbf{b}_\mu$. For the particular case $L_2 = 0$, equation (4.23) reduces to

$$L_{\mu\nu} = -L_{\nu\mu} = m_{ab} Q_\mu^{[a} Q_\nu^{b]}, \tag{4.23}$$

where m_{ab} is an antisymmetric $d \times d$ matrix of integers modulo $(d+2)$. It satisfies

$$\sum_{\mu=1}^{d+2} L_{\mu\nu} = 0. \tag{4.24}$$

The NC extension of equation (4.17) is given by the following algebra, to which we refer to as $\mathcal{A}_{\mathrm{nc}}(d+2)$:

$$\begin{aligned} &U_\mu U_\nu = \omega_{\mu\nu} \varpi_{\nu\mu} U_\nu U_\mu; \qquad && \mu, \nu = 1, \ldots, (d+1), \\ &U_\mu U_{d+2} = U_{d+2} U_\mu; && \mu = 1, \ldots, (d+1), \end{aligned} \tag{4.25}$$

where $\varpi_{\mu\nu}$ is the complex conjugate of $\omega_{\mu\nu}$. The latter are realized in terms of the Calabi–Yau charges data as follows:

$$\omega_{\mu\nu} = \exp \mathrm{i}\left(\frac{2\pi}{d+2} m_{ab} Q_\mu^a Q_\nu^b\right) = \omega^{m_{ab} Q_\mu^a Q_\nu^b}. \tag{4.26}$$

Using the property $\varpi_{\mu\nu}^{d+2} = 1$ and $\prod_{\mu}, \varpi_{\mu\nu} = 1$, one can check that the centre of the algebra (4.26) is given by

$$\mathcal{Z}(\mathcal{A}_{\mathrm{nc}}) = \lambda_1 U_1^{d+2} + \lambda_2 U_2^{d+2} + \cdots + \lambda_{d+1} U_{d+1}^{d+2} + \lambda_{d+2} I_{d+2} + \prod_{\mu=1}^{d+1} U_\mu. \tag{4.27}$$

The Schur lemma implies that this matrix equation can be written as

$$\mathcal{Z}(\mathcal{A}_{\mathrm{nc}}) = p(u_1, u_2, \ldots, u_{d+1}) I_{d+2}. \tag{4.28}$$

To determine the explicit expression of $p(u_1, u_2, \ldots, u_{d+1})$, let us discuss in what follows the matrix irreducible representations of the NC Calabi–Yau algebra for a regular point. In the next subsection we will give the representation for the fixed points, where the representation becomes reducible and corresponds to fractional branes.

Finite-dimensional representations of the algebra (4.26) are given by matrix subalgebras Mat$[n(d+2), C]$, the algebra of $n(d+2) \times n(d+2)$ complex matrices, with $n = 1, 2, \ldots$. Computing the determinant of both sides of equations (4.26)

$$\det(U_\mu U_\nu) = (\omega_{\mu\nu}\varpi_{\nu\mu})^D \det(U_\nu U_\mu) = \det(U_\nu U_\mu), \tag{4.29}$$

the dimension D of the representation to be such that

$$(\omega_{\mu\nu}\varpi_{\nu\mu})^D = 1. \tag{4.30}$$

Using the identity (4.19), one discovers that D is a multiple of $(d+2)$. We consider the fundamental $(d+2) \times (d+2)$ matrix representation obtained by introducing the following set $\{\mathbf{Q}; \mathbf{P}_{\alpha_{ab}}; a, b = 1, \ldots, d\}$ of matrices:

$$\mathbf{P}_{\alpha_{ab}} = \mathrm{diag}\big(1, \alpha_{ab}, \alpha_{ab}^2, \ldots, \alpha_{ab}^{d+1}\big); \qquad \mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & . & . & . & 1 \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 & 0 \\ 0 & 0 & 0 & . & . & 1 & 0 \end{pmatrix} \tag{4.31}$$

where $\alpha_{ab} = w^{m_{ab}}$ satisfying $\alpha_{ab}^{d+2} = 1$. From these expressions, it is not difficult to see that the $\{\mathbf{Q}; \mathbf{P}_{\alpha_{ab}}; a, b = 1, \ldots, d\}$ matrices obey the algebra

$$\mathbf{P}_\alpha \mathbf{P}_\beta = \mathbf{P}_{\alpha\beta}, \qquad \mathbf{P}_\alpha^{d+2} = 1, \qquad \mathbf{Q}^{d+2} = 1. \tag{4.32}$$

Using the identities

$$\mathbf{P}_{\alpha_\mu}^{n_\mu} \mathbf{Q}^{m_\mu} = \alpha_\mu^{n_\mu m_\mu} \mathbf{Q}^{m_\mu} \mathbf{P}_{\alpha_\mu}^{n_\mu}, \tag{4.33}$$

$$\big(\mathbf{P}_{\alpha_\mu}^{n_\mu} \mathbf{Q}^{m_\mu}\big)\big(\mathbf{P}_{\alpha_\nu}^{n_\nu} \mathbf{Q}^{m_\nu}\big) = \alpha_\mu^{n_i m_\nu} \alpha_\nu^{-m_\mu n_\nu} \big(\mathbf{P}_{\alpha_\nu}^{n_\nu} \mathbf{Q}^{m_\nu}\big)\big(\mathbf{P}_{\alpha_\mu}^{n_\nu} \mathbf{Q}^{m_\mu}\big), \tag{4.34}$$

one can check that the U_μ operators can be realized as

$$U_\mu = u_\mu \prod_{a,b=1}^{d} \big(\mathbf{P}_{\alpha_{ab}}^{Q_\mu^a} \mathbf{Q}^{Q\mu^b}\big), \tag{4.35}$$

where u_μ are C-number which should be thought of as in (4.17). From the Calabi–Yau condition, one can also check that the above representation satisfies

$$U_\mu^{d+2} = u_\mu^{d+2}\mathbf{I}_{d+2}, \qquad \prod_{\mu=1}^{d+1} U_\mu = \mathbf{I}_{d+2}\left(\prod_{\mu=1}^{d+1} u_\mu\right). \tag{4.36}$$

Putting these relations back into (4.29), one finds that the polynomial $p(u_\mu)$ is nothing but equation (4.17) of the Calabi–Yau hypersurface.

4.2.2. Solution for weighted projective spaces. In the case of weighted projective spaces with a weight vector $\delta = (\delta_1, \ldots, \delta_{d+2})$, the degree D of the Calabi–Yau polynomials and the corresponding N_i vertices are respectively given by equations (2.19)–(2.20) and (2.24)–(2.25). Note that integrality of the vertex entries requires that D should be the smallest common multiple of the weights δ_μ; that is $\frac{D}{\delta_\mu}$ an integer. Following the same reasoning as for the case of the projective space, one can work out a class of solutions of the constraint equations (4.11)–(4.13) in terms of powers of ω_μ. We get the result

$$\theta_{\mu\nu} = \exp \mathrm{i}2\pi\left[\frac{(\delta_\nu)L_{\mu\nu}}{D}\right], \tag{4.37}$$

where $L_{\mu\nu}$ is as in equation (4.23). Instead of being general, let us consider a concrete example dealing with the analogue of the quintic in the weighted projective space $WP^4_{\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\}}$. In this case the Calabi–Yau hypersurface $\sum_{\mu=1}^5 u_\mu^{\frac{D}{\delta_\mu}} + a_0\prod_{\mu=1}^5(u_\mu) = 0$, which for the example $\delta_1 = 2$ and $\delta_2 = \delta_3 = \delta_4 = \delta_5 = 1$ reduces to

$$u_1^3 + u_2^6 + u_3^6 + u_4^6 + u_5^6 + a_0\prod_{\mu=1}^5(u_\mu) = 0. \tag{4.38}$$

This polynomial has discrete isometries acting on the homogeneous coordinates u_μ as

$$u_\mu \to u_\mu\zeta_\mu^{\mathbf{a}_\mu^\nu} \qquad \mu = 1, \ldots, 5, \tag{4.39}$$

with $\zeta_1^3 = 1$ while $\zeta_\mu^6 = \omega^6 = 1$, i.e. $\zeta_1 = \omega^2$, and $\zeta_\mu = \omega$, and where the $\mathbf{a}_\mu^\nu$ are consistent with the Calabi–Yau condition. In the coordinate patch $\{u_\mu\}_{1\leqslant 4}$ with $u_5 = 1$, the equations defining the NC geometry of the Calabi–Yau (4.39) with discrete torsion, upon using the correspondence $u \to U$, are given by the algebra (5.1) where the $\theta_{\mu\nu}$ parameters should obey now the following constraint equations:

$$\begin{aligned}
&\theta_{\mu 1}^3 = 1, \qquad \mu = 2, 3, 4,\\
&\theta_{\mu\nu}^6 = 1, \qquad \nu \neq 1, \mu,\\
&\prod_{\mu=1}^4 \theta_{\nu\mu} = 1, \quad \forall\nu\\
&\theta_{\mu\nu}\theta_{\nu\mu} = 1, \quad \forall\mu, \nu.
\end{aligned} \tag{4.40}$$

Setting $\theta_{\mu\nu}$ as $\theta_{\mu\nu} = \omega^{L_{\mu\nu}}$ the constraints on $L_{\mu\nu}$ read

$$L_{\mu\nu} = -L_{\nu\mu} \text{ integers modulo 6}, \qquad L_{\mu 1} = \text{even modulo 6}. \tag{4.41}$$

Particular solutions of this geometry may be obtained by using antisymmetric bilinears of $\mathbf{a}_\mu^\nu$. Straightforward calculations show that, for $p_\mu = 1$, $L_{\mu\nu}$ is given by the following 4×4 matrix:

$$L_{\mu\nu} = \begin{pmatrix} 0 & k_1 - k_3 & -k_1 + k_2 & k_3 - k_2 \\ -k_1 + k_3 & 0 & k_1 & -k_3 \\ k_1 - k_2 & -k_1 & 0 & k_2 \\ -k_3 + k_2 & k_3 & -k_2 & 0 \end{pmatrix} \tag{4.42}$$

where the k_μ integers are such that $k_\mu - k_\nu \equiv 2r_{\mu\nu} \in 2Z$.

The NC algebra associated with equation (4.39) reads, in terms of $\omega_\mu = \omega^{k_\mu}$ and $\varpi_\mu = \omega^{-k_\mu}$,

$$\begin{aligned} U_1U_2 &= \omega_1\varpi_3U_2U_1, & U_1U_3 &= \varpi_1\omega_2U_3U_1, \\ U_1U_4 &= \omega_3\varpi_2U_4U_1, & U_2U_3 &= \omega_1U_3U_2, \\ U_2U_4 &= \varpi_3U_4U_2, & U_3U_4 &= \omega_2U_4U_3. \end{aligned} \tag{4.43}$$

Furthermore taking $\alpha = \omega_1\varpi_3$, $\beta = \omega_2\varpi_3$ and $\gamma = \omega_3$, one discovers an extension of the BL NC algebra (4.4); the difference is that now the deformation parameters are such that

$$\alpha^3 = \beta^3 = \gamma^6 = 1. \tag{4.44}$$

4.2.3. Fractional branes. Here we study the fractional branes corresponding to reducible representations at singular points. To illustrate the idea, we give a concrete example concerning the mirror geometry in terms of the $\mathbf{P}^{d+1}$ projective space. First note that the $\mathcal{A}_{\mathrm{nc}}(d+2)$ (4.37) corresponds to regular points of NC Calabi–Yau. This solution is irreducible and the branes do not fractionate. A similar solutions may be worked out as well for fixed points where we have fractional branes. We focus our attention on the orbifold of the *eight-tic*, namely,

$$u_1^8 + u_2^8 + \cdots + u_8^8 + a_0 \prod_{\mu=1}^{8} u_\mu = 0, \tag{4.45}$$

with the discrete isometries $\mathbf{Z}_8^6$ and Calabi–Yau charges Q_μ^a

$$\begin{aligned} Q_\mu^1 &= (1, -1, 0, 0, 0, 0, 0, 0), & Q_\mu^2 &= (1, 0, -1, 0, 0, 0, 0, 0) \\ Q_\mu^3 &= (1, 0, 0, -1, 0, 0, 0, 0), & Q_\mu^4 &= (1, 0, 0, 0, -1, 0, 0, 0) \\ Q_\mu^5 &= (1, 0, 0, 0, 0, -1, 0, 0), & Q_\mu^6 &= (1, 0, 0, 0, 0, 0, -1, 0). \end{aligned} \tag{4.46}$$

The corresponding NC algebra is deduced from the general one given in (4.26). At regular points, the matrix theory representation of this algebra is irreducible as shown in equations (4.37). However, the situation is more subtle at fixed points where representations are reducible. One way to deal with the singularity of the orbifold with respect to $\mathbf{Z}_8^6$ is to interpret the algebra as describing a $\mathbf{Z}_8^3$ orbifold with $\mathbf{Z}_8^3$ discrete torsions having singularities in codimension 4. Starting from equations (4.26) and choosing matrix coordinates U_5, U_6 and U_7 in the centre of the algebra by setting

$$(\omega_{\mu\nu}\varpi_{\nu\mu}) = 1, \qquad \text{for} \quad \mu = 5, 6, 7, 8; \quad \forall \nu = 1, \ldots, 8, \tag{4.47}$$

the algebra reduces to

$$\begin{aligned} U_1U_2 &= \alpha_1\alpha_2U_2U_1, & U_1U_3 &= \alpha_1^{-1}\alpha_3U_3U_1 \\ U_1U_4 &= \alpha_2^{-1}\alpha_3^{-1}U_4U_1, & U_2U_3 &= \alpha_1U_3U_2 \\ U_2U_4 &= \alpha_2U_4U_2, & U_3U_4 &= \alpha_3U_4U_3 \end{aligned} \tag{4.48}$$

and all remaining other relations are commuting. In these equations, the α_μ are such that $\alpha_\mu^8 = 1$; these are the phases $\mathbf{Z}_8^3$. At the singularity where the u_1, u_2, u_3 and u_4 moduli of equation (4.37) go to zero, one ends with the familiar result for orbifolds with discrete torsion. Therefore the D-branes fractionate in the codimension 4 singularities of the eight-tic geometry.

5. Link with the BL construction

In this section we want to rederive the results of [38] concerning NC quintic using the analysis developed in sections 3 and 4. Recall that in the coordinate patch $\{u_\mu\}_{1\leqslant 4}$ and $u_5 = 1$, the defining equations of NC geometry of the quintic with discrete torsion, upon using the correspondence $u \to U$, are given by the following operators algebra:

$$U_\mu U_\nu = \theta_{\mu\nu} U_\nu U_\mu, \qquad \nu > \mu = 1, \ldots, 4, \tag{5.1}$$

where the $\theta_{\mu\nu}$ are non-zero complex parameters. As the monomials U_μ^5 and $\prod_{\mu=1}^{5}(U_\mu)$ are commuting with all the U_μ, we also have

$$\left[U_\nu, U_\mu^5\right] = 0, \qquad \left[U_\nu, \prod_{\mu=1}^{4} U_\mu\right] = 0. \tag{5.2}$$

Compatibility between equations (5.1) and (5.2) gives constraint relations on $\theta_{\mu\nu}$, namely

$$\theta_{\nu\mu}^5 = 1, \tag{5.3}$$

$$\prod_{\mu=1}^{4} \theta_{\nu\mu} = 1, \quad \forall \nu \tag{5.4}$$

$$\theta_{\mu\nu}\theta_{\nu\mu} = 1; \qquad \theta_{\mu 5} = 1, \quad \forall \mu, \nu. \tag{5.5}$$

To establish the link between our way of doing and the construction of [40], it is interesting to note that the analysis of [40] corresponds in fact to a special representation of the formalism we developed so far. The idea is summarized as follows: first start from equation (3.1), which reads for the quintic as

$$u_\mu \to u_\mu \omega^{\mathbf{b}_\mu}, \tag{5.6}$$

where the $\mathbf{b}_\mu$ weights, $\mathbf{b}_\mu = \sum_{\nu=1}^{5} \mathbf{a}_\mu^\nu$, $\mu = 1, \ldots, 5$, are such that

$$\sum_{\nu=1}^{5} \mathbf{b}_\mu = 0. \tag{5.7}$$

This relation, interpreted as the Calabi–Yau condition, can be solved in different ways. A way to do this is to set the $\mathbf{b}_\mu$ weights as

$$\mathbf{b}_\mu = (p_1 + p_2 + p_3, -p_1, -p_2, -p_3, 0), \tag{5.8}$$

or equivalently by taking the weight components $\mathbf{b}_\mu^\nu$ as

$$\mathbf{a}_\mu^\nu = \begin{pmatrix} p_1 & p_2 & p_3 & 0 & 0 \\ -p_1 & 0 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 & 0 \\ 0 & 0 & -p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.9}$$

where p_a are integers modulo 5. More general solutions can be read from equations (4.23) by following the same method. The next step is to take $\theta_{\mu\nu} = \exp \mathrm{i}\left(\frac{2\pi}{5} L_{\mu\nu}\right)$ with $L_{\mu\nu}$ as follows:

$$L_{\mu\nu} = m_{12}\left(\mathbf{a}_\mu^1 \mathbf{a}_\nu^2 - \mathbf{a}_\nu^1 \mathbf{a}_\mu^2\right) - m_{23}\left(\mathbf{a}_\mu^2 \mathbf{a}_\nu^3 - \mathbf{a}_\nu^2 \mathbf{a}_\mu^3\right) + m_{13}\left(\mathbf{a}_\mu^1 \mathbf{a}_\nu^3 - \mathbf{a}_\nu^1 \mathbf{a}_\mu^3\right), \quad (5.10)$$

where $m_{12} = k_1, m_{23} = k_2$ and $m_{13} = k_3$ are integers modulo 5. For $p_\mu = 1$, we get

$$L_{\mu\nu} = \begin{pmatrix} 0 & k_1 - k_3 & -k_1 + k_2 & k_3 - k_2 \\ -k_1 + k_3 & 0 & k_1 & -k_3 \\ k_1 - k_2 & -k_1 & 0 & k_2 \\ -k_3 + k_2 & k_3 & -k_2 & 0 \end{pmatrix}, \quad (5.11)$$

and so the NC quintic algebra reads

$$\begin{aligned} U_1 U_2 &= \omega^{k_1 - k_3} U_2 U_1, & U_1 U_3 &= \omega^{-k_1 + k_2} U_3 U_1, \\ U_1 U_4 &= \omega^{k_3 - k_2} U_4 U_1, & U_2 U_3 &= \omega^{k_1} U_3 U_2, \\ U_2 U_4 &= \omega^{-k_3} U_4 U_2, & U_3 U_4 &= \omega^{k_2} U_4 U_3. \end{aligned} \quad (5.12)$$

Setting $\omega_\mu = \omega^{k_\mu}$ and $\varpi_\mu = \omega^{-k_\mu}$, the above relations become

$$\begin{aligned} U_1 U_2 &= \omega_1 \varpi_3 U_2 U_1, & U_1 U_3 &= \varpi_1 \omega_2 U_3 U_1, \\ U_1 U_4 &= \omega_3 \varpi_2 U_4 U_1, & U_2 U_3 &= \omega_1 U_3 U_2, \\ U_2 U_4 &= \varpi_3 U_4 U_2, & U_3 U_4 &= \omega_2 U_4 U_3. \end{aligned} \quad (5.13)$$

Now taking $\alpha = \omega_1 \varpi_3, \beta = \omega_2 \varpi_3$ and $\gamma = \omega_3$, one discovers exactly the BL algebra equations (4.8).

5.1. More on the NC quintic

As we mentioned, the solution given by equations (5.8) and (5.9) is in fact a special realization of the BL algebra (4.8). One can also write down other representations of the NC quintic; one of them is based on taking $\mathbf{a}_\mu^\nu$ as

$$\mathbf{a}_\mu^\nu = \begin{pmatrix} p_1 & 0 & p_3 & 0 & 0 \\ -2p_1 & p_2 & 0 & 0 & 0 \\ p_1 & -2p_2 & p_3 & 0 & 0 \\ 0 & p_2 & -2p_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.14)$$

The corresponding $\mathbf{b}_\mu$ weight vector is then

$$\mathbf{b}_\mu = (p_1 + p_3, -2p_1 + p_2, p_1 - 2p_2 + p_3, p_2 - 2p_3; 0), \quad (5.15)$$

with p_a are integers modulo 5. As one sees this is a different solution from that given in equations (5.8) and (5.9) as the corresponding Γ group of isometries acts differently on the u_μ variables leading then to a different orbifold with discrete torsion. Note that setting $p_\mu = 1$, the $\mathbf{a}_\mu^\nu$ weights are nothing but the Mori vectors of the blow up of the $\hat{A}_2$ affine singularity of $K3$, used in the geometric engineering method of 4D $\mathcal{N} = 2$ superconformal theories embedded in type II superstrings.

Setting $p_\mu = 1$ and using equations (5.10) and (5.14), the anti-symmetric $L_{\mu\nu}$ matrix reads

$$L_{\mu\nu} = \begin{pmatrix} 0 & k_1 + k_2 + 2k_3 & -2k_1 - 2k_2 & k_1 + k_2 - 2k_3 \\ -k_1 - k_2 - 2k_3 & 0 & 3k_1 - k_2 - 2k_3 & -2k_1 + 2k_2 + 4k_3 \\ 2k_1 + 2k_2 & -3k_1 + k_2 + 2k_3 & 0 & k_1 - 3k_2 - 2k_3 \\ -k_1 - k_2 + 2k_3 & 2k_1 - 2k_2 - 4k_3 & -k_1 + 3k_2 + 2k_3 & 0 \end{pmatrix} \quad (5.16)$$

where the k_1, k_2 and k_3 are integers modulo 5. The new algebra describing the NC quintic reads, in terms of the ω_μ and ϖ_ν generators of the Z_5^3, as

$$\begin{aligned} U_1U_2 &= \omega_1\omega_2\omega_3^2U_2U_1, & U_1U_3 &= \varpi_1^2\varpi_2^2U_3U_1, \\ U_1U_4 &= \omega_1\omega_2\varpi_3^2U_4U_1, & U_2U_3 &= \omega_1^3\varpi_2\varpi_3^2U_3U_2, \\ U_2U_4 &= \varpi_1^2\omega_2^2\omega_3^4U_4U_2, & U_3U_4 &= \omega_1\varpi_2^3\varpi_3^2U_4U_3. \end{aligned} \tag{5.17}$$

Setting $\alpha = \omega_1\omega_2\omega_3^2$, $\beta = \varpi_1\varpi_2\omega_3^2$ and $\gamma = \omega_1^2\varpi_2^2\varpi_3^4$, one discovers, once again, the BL algebra (4.7). Therefore equations (5.9) and (5.14) give two representations of the BL algebra.

5.2. *Comments on lower-dimensional CY manifolds*

The analysis we developed so far applies to complex d-dimensional homogeneous hypersurfaces with discrete torsion; $d \geqslant 2$. We have discussed the cases $d \geqslant 3$; here we want to complete this study for lower-dimensional Calabi–Yau manifolds, namely $K3$ and the elliptic curve. These are very special cases which deserve some comments. For the $K3$ surface in CP^3, we have

$$u_1^4 + u_2^4 + u_3^4 + u_3^4 + a_0 \prod_{\mu=1}^{4} u_\mu = 0. \tag{5.18}$$

This is a quartic polynomial with a $\mathbf{Z}_4 \times \mathbf{Z}_4$ symmetry acting on the u_i variables as

$$u_\mu \to w^{Q_\mu^a} u_\mu, \tag{5.19}$$

where $w^4 = 1$ and $\mathbf{a}_\mu^a$ are integers satisfying the Calabi–Yau condition $\sum_{\mu=1}^{4} Q_\mu^a = 0$. Choosing Q_μ^a as

$$Q_\mu^1 = (1, -1, 0, 0), \qquad Q_\mu^2 = (1, 0, -1, 0) \tag{5.20}$$

the 3×3 matrix $L_{\mu\nu}$ reads

$$L_{\mu\nu} = \begin{pmatrix} 0 & k & -k \\ -k & 0 & k \\ k & -k & 0 \end{pmatrix}. \tag{5.21}$$

Therefore the NC $K3$ algebra reads

$$\begin{aligned} U_1U_2 &= U_2U_1\,\mathrm{e}^{\mathrm{i}\frac{2\pi k}{4}}, & U_1U_3 &= U_3U_1\,\mathrm{e}^{-\mathrm{i}\frac{2\pi k}{4}}, & U_1U_4 &= U_4U_1, \\ U_2U_3 &= U_3U_2\,\mathrm{e}^{\mathrm{i}\frac{2\pi k}{4}}, & U_2U_4 &= U_4U_2, & U_3U_4 &= U_4U_3, \end{aligned} \tag{5.22}$$

where k is an integer modulo 4. Note that one gets similar results by making other choices of Q_i^a such as

$$Q_\mu^1 = (1, -2, 1, 0), \qquad Q_\mu^2 = (1, 1, -2, 0). \tag{5.23}$$

More general results may also be written down for $K3$ embedded in $WP_{(\delta_1,\delta_2,\delta_3,\delta_4)}$. In the case of a one-dimensional elliptic fibre given by a cubic in P^2

$$u_1^3 + u_2^3 + u_3^3 + a_0 \prod_{\mu=1}^{3} u_\mu = 0, \tag{5.24}$$

the constraint equations defining non-commutativity are trivially solved. They show that $L_{\mu\nu} = 0$ and so $\theta_{12} = 1$ leading then to a commutative geometry. NC geometries involving elliptic curves can be constructed; the idea is to consider orbifolds of products of elliptic curves. More details are exposed in the following section. Related ideas with fractional branes will be considered as well.

6. NC elliptic manifolds

In this section we want to refine the study of the NC Calabi–Yau hypersurface defined in terms of orbifolds of elliptic curves. The original idea of this construction was introduced first in [38], see also [72], in connection with the NC orbifold $\frac{T^6}{Z_2^2}$. The method is quite similar to that discussed for the quintic and generalized Calabi–Yau geometries in sections 4 and 5. To start, consider the following elliptic realization of $\frac{T^{2n+2}}{\Gamma}$, that is T^{2n+2} is represented by the product of $(n+1)$ elliptic curves $(T^2)^{\otimes(2k+1)}$ where $n = 2k$. Each elliptic curve is given in Weierstrass form as

$$y_\mu^2 = x_\mu(x_\mu - 1)(x_\mu - a_\mu), \qquad \mu = 1, \ldots, n+1, \tag{6.1}$$

with a point added at infinity $\mu = 1, \ldots, n+1$. The system $\{(x_\mu, y_\mu); \mu = 1, \ldots, n+1\}$ defines the complex coordinates of C^{2n+2} space and a_μ are $(n+1)$ complex moduli. For later use, we introduce the algebra $\mathcal{A}_c$ of holomorphic functions on T^{2n+2}. This is a commutative algebra generated by monomials in the x_μ and y_μ with conditions (6.1). The discrete group Γ acts on x_μ and y_μ as

$$x_\mu \to x'_\mu = x_\mu, \qquad y_\mu \to y'_\mu = y_\mu \omega^{Q_\mu}, \tag{6.2}$$

where ω is an element of the discrete group Γ and where Q_μ are integers which should be compared with equation (4.24). Note that if one requires equations (6.1) to be invariant under Γ, then ω^2 should be equal to one that is $\omega = \pm 1$. If one requires moreover that the monomial $\prod_{\mu=1}^{n+1} y_\mu$ or again the holomorphic $((n+1), 0)$ form $\mathrm{d}y_1 \wedge \mathrm{d}y_2 \cdots \wedge \mathrm{d}y_{n+1}$, to be invariants under the orbifold action, it follows then that $\prod_{\mu=1}^{n+1} \omega^{Q_\mu} = \omega^{\sum_\mu Q_\mu} = 1$. This is equivalent to

$$\sum_{\mu=1}^{n+1} Q_\mu = 0, \quad \text{modulo } 2, \tag{6.3}$$

which defines the Calabi–Yau condition for the orbifold $\frac{T^{2n+2}}{\Gamma}$. Therefore the Γ discrete group is given by $\Gamma = (\mathbf{Z}_2)^{\otimes n}$. Following the discussion we made in section 4, this equation can also be rewritten as

$$\sum_{\mu=1}^{n+1} Q_\mu^a = 0, \qquad \text{modulo } 2; \quad a = 1, \ldots, n. \tag{6.4}$$

The four fixed points of the orbifold for each two torus T^2 are located at $(x_\mu = 0, 1, a_\mu; y_\mu = 0)$ and the point at infinity, i.e. $(x_\mu = \infty; y_\mu = \infty)$. The latter can be brought to a fixed finite point by working in another coordinate patch related to the old one by using the change of variables:

$$y_\mu \to y'_\mu = \frac{y_\mu}{x_\mu^2}, \qquad x_\mu \to x'_\mu = \frac{1}{x_\mu}. \tag{6.5}$$

The NC version of the orbifold $\frac{T^{2n+2}}{\Gamma}$ is obtained by substituting the usual commuting x_μ and y_μ variables by the matrix operators X_μ and Y_μ respectively. These matrix operators satisfy the following NC algebra structure:

$$Y_\mu Y_\nu = \theta_{\mu\nu} Y_\nu Y_\mu, \tag{6.6}$$

$$X_\mu X_\nu = X_\nu X_\mu, \tag{6.7}$$

$$X_\mu Y_\nu = Y_\nu X_\mu, \tag{6.8}$$

with

$$Y_\mu Y_\nu^2 = Y_\nu^2 Y_\mu, \tag{6.9}$$

as is required by equation (6.1) and

$$\left[Y_\mu, \prod_{\nu=1}^{n+1} Y_\nu\right] = 0. \tag{6.10}$$

As for the case of the homogeneous hypersurfaces we considered in sections 4 and 5, here also the Calabi–Yau condition is fulfilled by imposing that the $\prod_{\nu=1}^{n+1} Y_\nu$ belongs to the centre of the NC algebra $\mathcal{A}_{\rm nc}$. Now using equations (6.6)–(6.10), one gets the explicit expression of the $\theta_{\mu\nu}$ by solving the following constraint equations:

$$\theta_{\mu\nu}\theta_{\mu\nu} = 1, \tag{6.11}$$

$$\prod_{\mu=1}^{n+1} \theta_{\mu\nu} = 1, \tag{6.12}$$

$$\theta_{\mu\nu}\theta_{\nu\mu} = 1, \qquad \theta_{\mu\mu} = 1. \tag{6.13}$$

Note that equation (6.11) is a strong constraint which will have a drastic consequence on the solving of non-commutativity. Comparing this relation to equation (6.12), one can write

$$\begin{aligned} &\theta_{\mu\nu} = (-1)^{L_{\mu\nu}}, \\ &\sum_{\mu=1}^{n+1} L_{\mu\nu} = 0, \quad \text{modulo } 2, \end{aligned} \tag{6.14}$$

where $L_{\mu\nu}$ is the antisymmetric matrix, $L_{\mu\nu} = -L_{\nu\mu}$, of integer entries given by

$$L_{\mu\nu} = \Omega_{ab} Q_\mu^a Q_\nu^b; \tag{6.15}$$

where $\Omega_{ab} = -\Omega_{ba}$, and $\Omega_{ab} = 1$ for $a < b$. This relation should be compared to equation (4.25). Moreover, one learns from equation (6.14) that the two cases should be distinguished. The first one corresponds to the case $\theta_{\mu\nu} = -1 \forall \mu \neq \nu$, that is,

$$L_{\mu\nu} = 1; \quad \text{modulo } 2. \tag{6.16}$$

In this case, the constraint (6.12) is fulfilled provided n is even; i.e. $n = 2k$. So the group Γ is given by $\Gamma = (\mathbf{Z}_2)^{\otimes 2k}$. The second case corresponds to the situation where some $\theta_{\mu\nu}$ are equal to 1:

$$L_{\mu\nu} = 1; \quad \text{modulo } 2; \quad \mu = 1, \ldots, (n+1-r); \quad \mu \neq \nu \tag{6.17}$$

$$L_{\mu\nu} = 0; \quad \text{modulo } 2; \quad \mu = (n-r+2), \ldots, n+1, \tag{6.18}$$

where we have rearranged the variables so that the matrix takes the form

$$L_{\mu\nu} = \begin{pmatrix} L'_{\mu'\nu'} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{6.19}$$

In this case equation (6.17) shows that n is even if r is even and odd if r is odd. In what follows we build the solutions of the NC algebra (6.6)–(6.8) using finite-dimensional matrices.

6.1. Solution I

Putting relation (6.16) back into equations (6.6)–(6.8), the non-commutativity algebra, which reads

$$Y_\mu Y_\nu = -Y_\nu Y_\mu, \tag{6.20}$$

$$Y_\mu Y_\nu^2 = Y_\nu^2 Y_\mu \tag{6.21}$$

$$X_\mu X_\nu = X_\nu X_\mu, \tag{6.22}$$

$$X_\mu Y_\nu = Y_\nu X_\mu, \tag{6.23}$$

may be realized in terms of $2^k \times 2^k$ matrices of the space of matrices $M(2^k, C)$. These are typical relations naturally solved by using the $2k$-dimensional Clifford algebra generated by the basis system $\{\Gamma^\iota, \mu = 1, \dots, 2k\}$:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}, \qquad \{\Gamma^i, \Gamma^{2k+1}\} = 0, \tag{6.24}$$

where $\Gamma^{2k+1} = \prod_{i=1}^{2k} \Gamma^i$. We therefore have

$$Y_\mu = b_\mu \Gamma^\mu, \qquad \mu = 1, \dots, 2k, \tag{6.25}$$

$$Y_{2k+1} = b_0 \Gamma^{2k+1}, \tag{6.26}$$

$$X_\mu = x_\mu I_{2^k}, \tag{6.27}$$

where the b_μ are complex scalars. This solution has remarkable features: (i) after choosing a Hermitian Γ matrices representation, one can see at the fixed planes, where $2k$ variables among the $(2k + 1)$ y_μ act by zero and all others zero, that there exists a multiplicity of inequivalent representations for each set of roots x_μ of the Weierstrass forms. Therefore, one can get 2^k distinct NC points, as there are 2^k irreducible representations corresponding to 2^k eigenvalues of the non-zero matrix variable and so the branes fractionate on the singularity. (ii) The non-commutative points of the singular planes are then seen to be a $2k$ cover of the commutative singular plane, which is a $(CP^1)^{\otimes k}$. The $2k$ cover is branched around the four points $x_k = 0, 1, a_k, \infty$ and hence the NC points form an elliptic manifold T^{2k} of the form equation (6.1). Around each of these four points, there is a $(\mathbf{Z}_2)$ monodromy of the representations, which is characteristic of the local singularity as measuring the effect of discrete torsion.

6.2. Solution II

Putting relations (6.17) back into equations (6.6)–(6.8), the resulting NC algebra depends on the integer r and reads

$$Y_\mu Y_\nu = -Y_\nu Y_\mu, \qquad \mu, \nu = 1, \dots, (n + 1 - r). \tag{6.28}$$

$$Y_\mu Y_\nu = Y_\nu Y_\mu, \qquad \mu, \nu = (n + 2 - r), \dots, (n + 1), \tag{6.29}$$

$$Y_\mu Y_\nu^2 = Y_\nu^2 Y_\mu, \qquad \mu = 1, \dots, (n + 1), \tag{6.30}$$

$$X_\mu X_\nu = X_\nu X_\mu, \tag{6.31}$$

$$X_\mu Y_\nu = Y_\nu X_\mu. \tag{6.32}$$

For $r = 2s$ even, irreducible representations of this algebra are given, in terms of $2^{k-s} \times 2^{k-s}$ matrices of the space $M(2^{k-s}, C)$, by the $2(k - s)$-dimensional Clifford algebra. The result is

$$Y_\mu = b_\mu \Gamma^\mu, \qquad i = 1, \ldots, 2(k-s), \tag{6.33}$$

$$Y_{2(k-s)+1} = b_0 \prod_{\mu=1}^{2(k-s)} \Gamma^\mu, \tag{6.34}$$

$$Y_\mu = y_\mu I_{2^{k-s}}, \qquad i = 2(k-s+1), \ldots, (2k+1), \tag{6.35}$$

$$X_\mu = x_\mu I_{2^{k-s}}. \tag{6.36}$$

At the end of this section, we would like to note that this analysis could be extended to a general case initiated in [72], where the elliptic curves are replaced by $K3$ surfaces. This might be applied to the resolution of orbifold singularities in the moduli space of certain models, describing a D2-brane wrapped n times over the fibre of an elliptic $K3$, as follows [73]

$$\mathcal{M}_{1,n} = \mathrm{Sym}(K3) = \frac{K3^{\otimes n}}{S_n}. \tag{6.37}$$

Here $\mathcal{M}_{1,n}$ denotes the moduli space of a D2-brane with charges $(1, n)$ and S_n is the group of permutation of n elements.

7. Conclusion

In this paper we have studied the NC version of Calabi–Yau hypersurface orbifolds using the algebraic geometry approach of [39, 40] combined with toric geometry method of complex manifolds. Actually this study extends the analysis of the NC Calabi–Yau manifolds with discrete torsion initiated in [40] and exposes explicitly the solving of non-commutativity in terms of toric geometry data. Our main results may be summarized as follows:

(1) First we have developed a method of getting d complex Calabi–Yau mirror coset manifolds $C^{k+1}/C^{*r}, k-r = d$, as hypersurfaces in WP^{d+1}. The key idea is to solve the y_i invariants (2.12) and (2.13) of mirror geometry in terms of invariants of the C^* action of the weighted projective space and the toric data of C^{k+1}/C^{*r}. As a matter of fact, the above-mentioned mirror Calabi–Yau spaces are described by homogeneous polynomials $P_\Delta(u)$ of degree $D = \sum_{\mu=1}^{d+2} \delta_\mu = \sum_{\mu=1}^{d+2} \sum_{a=1}^{r} p_a q_\mu^a$, where δ_μ are projective weights of the C^* action, q_μ^a are entries of the well-known Mori vectors and the p_a are given integers. Then we have determined the general group Γ of discrete isometries of $P_\Delta(u)$. We have shown by explicit computation that in general one should distinguish two cases Γ_0 and Γ_{cd}. First Γ_0 is the group of isometries of the hypersurface $\sum_{\mu=1}^{d+2} u_\mu^{D/\delta_\mu} + a_0 \prod_{\mu=1}^{d+2} (u_\mu) = 0$, generated by the changes $u'_\mu = u_\mu (\mathcal{W})^{\mathbf{b}_\mu}$, where the weight vector $\mathbf{b}_\mu$ is given by the sum of Q_μ and ξ_μ respectively associated with the Calabi–Yau charges and the vertices data of the toric manifold M_Δ^{d+1}. In case where the complex deformations are taken into account (see equation (2.27)), the symmetry group reduces to the subgroup Γ_{cd} generated by the changes $u'_\mu = u_\mu (\mathcal{W})^{\mathbf{b}_\mu}$ where now $\mathbf{b}_\mu$ has no ξ_μ factor.

(2) Using the above results and the algebraic geometry approach, we have developed a method of building NC Calabi–Yau orbifolds in toric manifolds. Non-commutativity is solved in terms of the discrete torsion and bilinears of the weight vector $\mathbf{a}_\mu^\nu$; see equation (3.11). Among our results, we have worked out several matrix representations of the NC quintic algebra obtained in [40] by using various Calabi–Yau toric geometry data. We have also given the generalization of these results to higher-dimensional Calabi–Yau hypersurface orbifolds and derived the explicit form of the non-commutative D-tic orbifolds.

(3) We have extended to higher complex dimensional Calabi–Yau's realized as toric orbifold of type $\frac{T^{4k+2}}{\Gamma}$ with discrete torsion. Due to constraint equations on non-commutativity, we have shown that in the elliptic realization of the two torii factors, Γ is constrained to be equal to $\mathbf{Z}_2^{2k}$, the real dimension should be $2k+2$ and non-commutativity is solved in terms of the $2k$-dimensional Clifford algebra. We have also discussed the fractional brane which corresponds to reducible representations of toric Calabi–Yau algebras.

Acknowledgments

One of us (AB) is very grateful to J McKay and A Sebbar for discussions and scientific helps. He would also like to thank J J Manjarín and P Resco for earlier collaboration on this subject. This work is partially supported by PARS, programme de soutien à la recherche scientifique; Université Mohammed V-Agdal, Rabat.

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